4.4 METHOD OF UNDETERMINED COEFFICIENTS

TO SOLVE $y''+py'+qy=g \leftarrow$ g IS CALLED THE FORCING TERM

① SOLVE $y''+py'+qy=0$ + GET $y_h = C_1y_1 + C_2y_2$

② FIND $\underset{\substack{\uparrow \\ \text{PARTICULAR}}}{1}$ SOL'N OF $y''+py'+qy=g$ + GET y_p

③ $y = C_1y_1 + C_2y_2 + y_p$ IE. $y_h + y_p$

FOR NON-HOMOGENEOUS 2^{ND} ORDER LDE WITH CONSTANT COEF'S

START WITH AN EDUCATED GUESS FOR y_p

eg. $y''-y'+9y=3\sin 3x$

$r^2-r+9=0 \longrightarrow r=\dfrac{1\pm\sqrt{1-36}}{2} = \dfrac{1}{2}\pm\dfrac{\sqrt{35}}{2}i$

$y_h = C_1e^{\frac{1}{2}x}\sin\frac{\sqrt{35}}{2}x + C_2e^{\frac{1}{2}x}\cos\frac{\sqrt{35}}{2}x$

GUESS $y_p = A\cos 3x + B\sin 3x$

$y_p' = 3B\cos 3x - 3A\sin 3x$

$y_p'' = -9A\cos 3x - 9B\sin 3x$

$$y_p''-y_p'+9y_p = \begin{pmatrix}\cancel{-9A}\\ -3B\\ \cancel{+9A}\end{pmatrix}\cos 3x + \begin{pmatrix}\cancel{-9B}\\ +3A\\ \cancel{+9B}\end{pmatrix}\sin 3x$$

$= -3B\cos 3x + 3A\sin 3x = 3\sin 3x$

$-3B=0 \qquad 3A=3$

$y = \cos 3x + C_1e^{\frac{1}{2}x}\sin\frac{\sqrt{35}}{2}x + C_2e^{\frac{1}{2}x}\cos\frac{\sqrt{35}}{2}x \leftarrow y_p = \cos 3x \leftarrow B=0 \qquad A=1$

SOLVE $z'' - 6z' + 9z = x^2 + e^x \longrightarrow r^2 - 6r + 9 = 0 \longrightarrow r = 3, 3$

$z_h = c_1 e^{3x} + c_2 x e^{3x}$

$z'' - 6z' + 9z = x^2$

GUESS $z_p = Ax^2 + Bx + C$

$z_p' = 2Ax + B$

$z_p'' = 2A$

$\longrightarrow$ ~~$z_p'' - 6z_p' + 9z_p = 2A - 12Ax + 9Ax^2 = x^2$~~

~~IMPOSSIBLE $\begin{cases} 9A = 1 \rightarrow A = \frac{1}{9} \\ -12A = 0 \rightarrow A = 0 \\ 2A = 0 \rightarrow A = 0 \end{cases}$~~

$z_p'' - 6z_p' + 9z_p$

$= 2A$

$-12Ax - 12B$

$9Ax^2 + 9Bx + 9C$

$= 9Ax^2 + (-12A + 9B)x + (2A - 12B + 9C) = x^2$

$9A = 1$ $\quad -\cancel{12}^{4}\left(\frac{1}{\cancel{9}_{3}}\right) + 9B = 0$ $\quad 2\left(\frac{1}{9}\right) - \cancel{12}^{4}\left(\frac{4}{\cancel{27}_{9}}\right) + 9C = 0$

$A = \frac{1}{9}$ $\quad 9B = \frac{4}{3}$ $\quad 9C = \frac{14}{9}$

$B = \frac{4}{27}$ $\quad C = \frac{14}{81}$

$z_p = \frac{1}{9}x^2 + \frac{4}{27}x + \frac{14}{81}$

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IF y_{p_1} IS A SOL'N OF $y'' + py' + qy = g_1$

AND y_{p_2} IS A SOL'N OF $y'' + py' + qy = g_2$

THEN $y_{p_1} + y_{p_2}$ IS A SOL'N OF $y'' + py' + qy = g_1 + g_2$

IF $L[y] = y'' + py' + qy$

$L[y_{p_1}] = g_1 \quad L[y_{p_2}] = g_2$

$L[y_{p_1} + y_{p_2}] = L[y_{p_1}] + L[y_{p_2}]$
$= g_1 + g_2$

$z'' - 6z' + 9z = e^x$

GUESS $z_p = Ae^x$

$z_p' = Ae^x$

$z_p'' = Ae^x$

$z_p'' - 6z_p' + 9z_p = (1 - 6 + 9)Ae^x$

$= 4Ae^x = e^x$

$4A = 1$

$A = \frac{1}{4}$

$z_p = \frac{1}{4}e^x$

FOR $z'' - 6z' + 9z = x^2 + e^x$

$z_p = \frac{1}{9}x^2 + \frac{4}{27}x + \frac{14}{81} + \frac{1}{4}e^x$

$z = \frac{1}{9}x^2 + \frac{4}{27}x + \frac{14}{81} + \frac{1}{4}e^x + C_1e^{3x} + C_2xe^{3x}$

$$y'' - y = e^x \longrightarrow y'' - y = 0 \longrightarrow r^2 - 1 = 0$$

$$r = 1, -1$$

$$y_h = \underline{c_1 e^x} + c_2 e^{-x}$$

~~GUESS~~ $\cancel{y_p = \underline{Ae^x}}$ ← BAD GUESS SHARES A TERM WITH y_h

$$\cancel{y_p' = Ae^x}$$

$$\cancel{y_p'' = Ae^x}$$

$$\cancel{y_p'' - y_p = Ae^x - Ae^x = 0 = e^x}$$

~~IMPOSSIBLE~~

GUESS $y_p = Axe^x$

$$y_p' = Ae^x + Axe^x = A(1+x)e^x$$

$$y_p'' = Ae^x + A(1+x)e^x = A(2+x)e^x$$

$$\begin{aligned} y_p'' - y &= A(2+x)e^x \\ &\quad - A(x)e^x \\ &= A(2)e^x \\ &= 2Ae^x = e^x \end{aligned}$$

$$2A = 1$$

$$A = \tfrac{1}{2}$$

$$y_p = \tfrac{1}{2}xe^x$$

$$y = \tfrac{1}{2}xe^x + c_1 e^x + c_2 e^{-x}$$

$$y'' + y = \sin x \rightarrow r^2 + 1 = 0 \rightarrow r = \pm i \longrightarrow y_h = c_1 \sin x + c_2 \cos x$$

$$y_p = (A \sin x + B \cos x)x = Ax \sin x + Ax \cos x$$